Module 4: Stress-Strain Relations

4.2.1 ELASTIC STRAIN ENERGY FOR UNIAXIAL STRESS

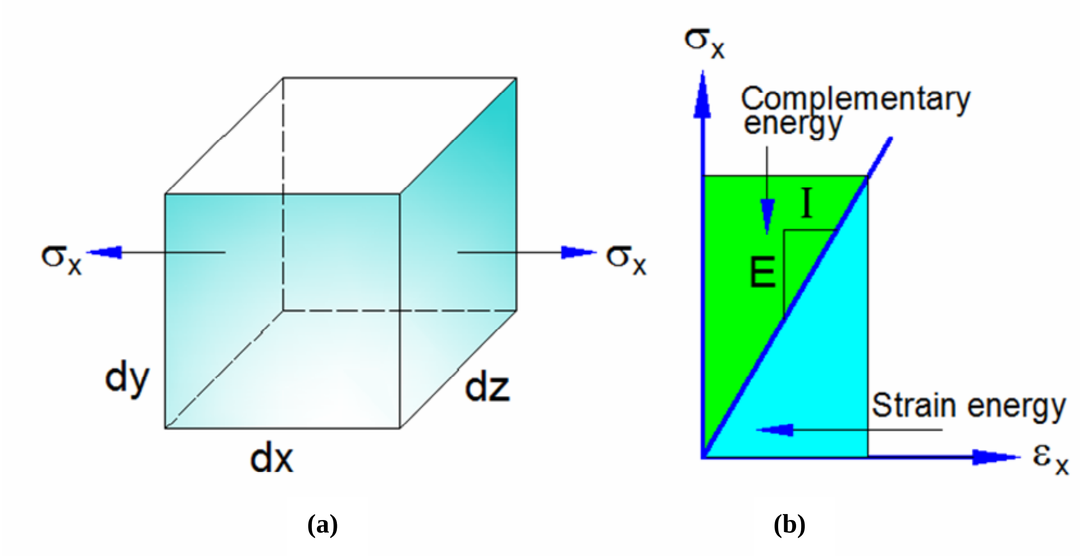


Figure 4.1 Element subjected to a Normal stress

In mechanics, energy is defined as the capacity to do work, and work is the product of force and the distance, in the direction, the force moves. In solid deformable bodies, stresses multiplied by their respective areas, results in forces, and deformations are distances. The product of these two quantities is the internal work done in a body by externally applied forces. This internal work is stored in a body as the internal elastic energy of deformation or the elastic strain energy.

Consider an infinitesimal element as shown in Figure 4.1a, subjected to a normal stress σ_x. The force acting on the right or the left face of this element is $\sigma_x\, dydz$. This force causes an elongation in the element by an amount $\varepsilon_x\, dx$, where ε_x is the strain in the direction x. The average force acting on the element while deformation is taking place is $\sigma_x \frac{dy\,dz}{2}$.

This average force multiplied by the distance through which it acts is the work done on the element. For a perfectly elastic body no energy is dissipated, and the work done on the element is stored as recoverable internal strain energy. Therefore, the internal elastic strain energy U for an infinitesimal element subjected to uniaxial stress is

$$dU = \frac{1}{2}\sigma_x\, dy\, dz \;\times \varepsilon_x\, dx$$

$= \frac{1}{2}\ \sigma_x\, \varepsilon_x\, dx\ dy\ dz$

Therefore, $dU = \frac{1}{2}\sigma_x\ \varepsilon_x\ dV$

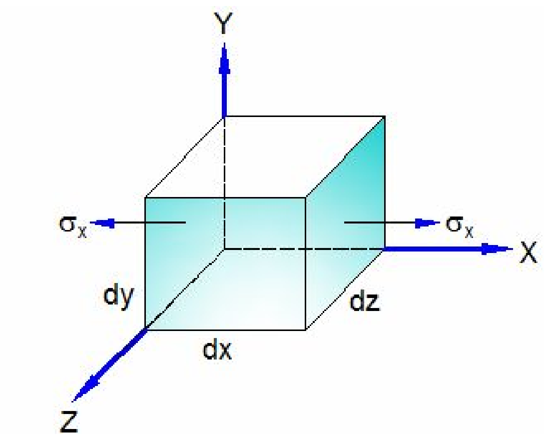
where dV = volume of the element.

Thus, the above expression gives the strain energy stored in an elastic body per unit volume of the material, which is called strain-energy density U_0.

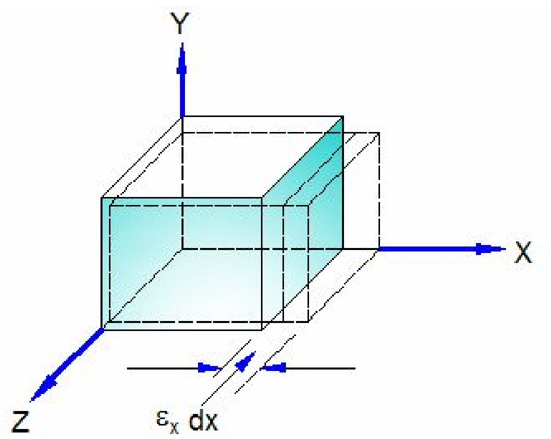
Hence, $\frac{dU}{dV} = U_0 = \frac{1}{2}\sigma_x\ \varepsilon_x$

The above expression may be graphically interpreted as an area under the inclined line on the stress-strain diagram as shown in Figure (4.1b). The area enclosed by the inclined line and the vertical axis is called the complementary energy. For linearly elastic materials, the two areas are equal.

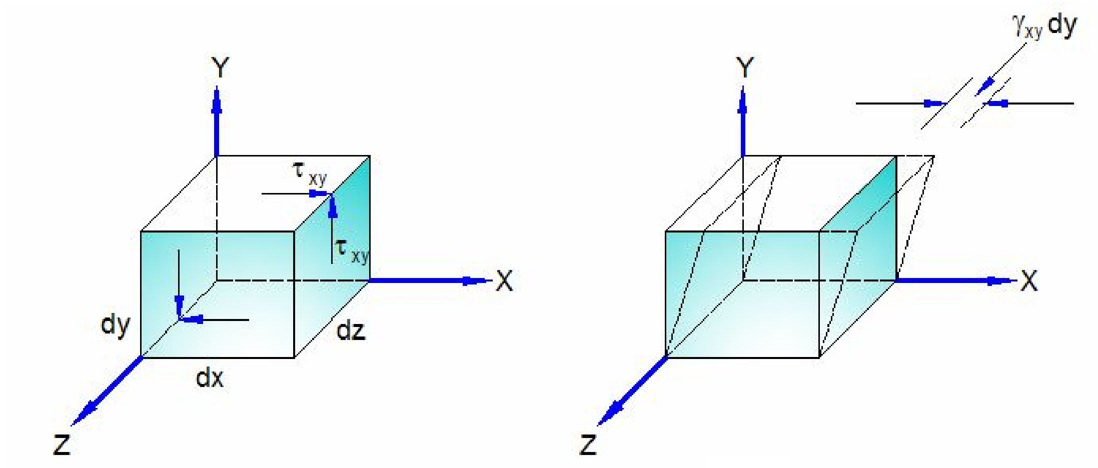
4.2.2 STRAIN ENERGY IN AN ELASTIC BODY



(a)



(b)



(c) (d)

Figure 4.2. Infinitesimal element subjected to: uniaxial tension (a), with resulting deformation (b); pure shear (c), with resulting deformation (d)

When work is done by an external force on certain systems, their internal geometric states are altered in such a way that they have the potential to give back equal amounts of work whenever they are returned to their original configurations. Such systems are called conservative, and the work done on them is said to be stored in the form of potential energy. For example, the work done in lifting a weight is said to be stored as a gravitational potential energy. The work done in deforming an elastic spring is said to be stored as elastic potential energy. By contrast, the work done in sliding a block against friction is not recoverable; i.e., friction is a non-conservative mechanism.

Now we can extend the concept of elastic strain energy to arbitrary linearly elastic bodies subjected to small deformations.

Figure 4.2(a) shows a uniaxial stress component σ_x acting on a rectangular element, and Figure 4.2(b) shows the corresponding deformation including the elongation due to the strain component ε_x. The elastic energy stored in such an element is commonly called strain energy.

In this case, the force $\sigma_x\ dydz$ acting on the positive x face does work as the element undergoes the elongation $\varepsilon_x dx$. In a linearly elastic material, strain grows in proportion to stress. Thus the strain energy dU stored in the element, when the final values of stress and strain are σ_x and ε_x is

$$dU = \frac{1}{2}\ (\sigma_x\ dydz)\ (\varepsilon_x dx)$$

$$= \frac{1}{2}\,\sigma_x\,\varepsilon_x\,dV \qquad (4.28)$$

where $dV = dx\,dy\,dz$ = volume of the element.

If an elastic body of total volume V is made up of such elements, the total strain energy U is obtained by integration

$$U = \frac{1}{2}\int_v \sigma_x\,\varepsilon_v\,dV \qquad (4.29)$$

Taking $\sigma_x = \dfrac{P}{A}$ *and* $\varepsilon_x = \dfrac{\delta}{L}$

where P = uniaxial load on the member

δ = displacement due to load P

L = length of the member,

A = cross section area of the member

We can write equation (4.28) as

$$U = \frac{1}{2}\left(\frac{P}{A}\right)\left(\frac{\delta}{L}\right)\int_v dV$$

Therefore, $U = \dfrac{1}{2}P.\delta$ since $V = L \times A$ (4.30)

Next consider the shear stress component τ_{xy} acting on an infinitesimal element in Figure 4.2(c). The corresponding deformation due to the shear strain component γ_{xy} is indicated in Figure 4.2(d). In this case the force $\tau_{xy}\,dxdz$ acting on the positive y face does work as that face translates through the distance $\gamma_{xy}\,dy$. Because of linearity, γ_{xy} and τ_{xy} grow in proportion as the element is deformed.

The strain energy stored in the element, when the final values of strain and stress are γ_{xy} and τ_{xy} is

$$dU = \frac{1}{2}\left(\tau_{xy}\,dxdz\right)\left(\gamma_{xy}\,dy\right)$$

$$= \frac{1}{2}\tau_{xy}\gamma_{xy}\,dxdydz$$

Therefore, $dU = \dfrac{1}{2}\,\tau_{xy}\,\gamma_{xy}\,dV$ (4.31)

The results are analogous to equation (4.28) and equation (4.31) can be written for any other pair of stress and strain components (for example, σ_y and ε_y or τ_{yz} and γ_{yz}) whenever the stress component involved is the only stress acting on the element.

Finally, we consider a general state of stress in which all six stress components are present. The corresponding deformation will in general involve all six strain components. The total strain energy stored in the element when the final stresses are σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} and the final strains are ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{zx} is thus

$$dU = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right) dV \qquad (4.32)$$

In general, the final stresses and strains vary from point to point in the body. The strain energy stored in the entire body is obtained by integrating equation (4.32) over the volume of the body.

$$U = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right) dV \qquad (4.33)$$

The above formula for strain energy applies to small deformations of any linearly elastic body.

4.2.3 BOUNDARY CONDITIONS

The boundary conditions are specified in terms of surface forces on certain boundaries of a body to solve problems in continuum mechanics. When the stress components vary over the volume of the body, they must be in equilibrium with the externally applied forces on the boundary of the body. Thus the external forces may be regarded as a continuation of internal stress distribution.

Consider a two dimensional body as shown in the Figure 4.3

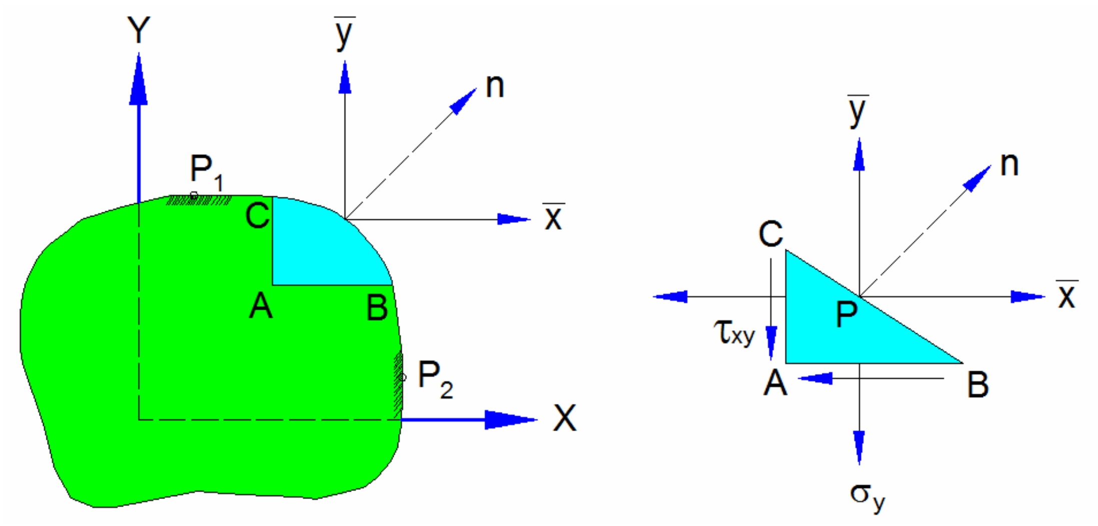


Figure 4.3 An element at the boundary of a body

Take a small triangular prism *ABC*, so that the side BC coincides with the boundary of the plate. At a point *P* on the boundary, the outward normal is n. Let $\overline{X}$ and $\overline{Y}$ be the components of the surface forces per unit area at this point of boundary. $\overline{X}$ and $\overline{Y}$ must be a continuation of the stresses σ_x, σ_y and τ_{xy} at the boundary. Now, using Cauchy's equation, we have

$$T_x = \overline{X} = \sigma_x l + \tau_{xy} m \qquad \text{(a)}$$

$$T_y = \overline{Y} = \tau_{xy} l + \sigma_y m$$

in which *l* and *m* are the direction cosines of the normal *n* to the boundary.

For a particular case of a rectangular plate, the co-ordinate axes are usually taken parallel to the sides of the plate and the boundary conditions (equation a) can be simplified. For example, if the boundary of the plate is parallel to *x*-axis, at point P_1, then the boundary conditions become

$$\overline{X} = \tau_{xy} \text{ and } \overline{Y} = \sigma_y \qquad \text{(b)}$$

Further, if the boundary of the plate is parallel to y-axis, at point P_2, then the boundary conditions become

$$\overline{X} = \sigma_x \text{ and } \overline{Y} = \tau_{xy} \qquad \text{(c)}$$

It is seen from the above that at the boundary, the stress components become equal to the components of surface forces per unit area of the boundary.

4.2.4 ST. VENANT'S PRINCIPLE

For the purpose of analysing the statics or dynamics of a body, one force system may be replaced by an equivalent force system whose force and moment resultants are identical. Such force resultants, while equivalent need not cause an identical distribution of strain, owing to difference in the arrangement of forces. St. Venant's principle permits the use of an equivalent loading for the calculation of stress and strain.

St. Venant's principle states that if a certain system of forces acting on a portion of the surface of a body is replaced by a different system of forces acting on the same portion of the body, then the effects of the two different systems at locations sufficiently far distant from the region of application of forces, are essentially the same, provided that the two systems of forces are statically equivalent (i.e., the same resultant force and the same resultant moment).

St. Venant principle is very convenient and useful in obtaining solutions to many engineering problems in elasticity. The principle helps to the great extent in prescribing the boundary conditions very precisely when it is very difficult to do so. The following figures 4.4, 4.5 and 4.6 illustrate St. Venant principle.

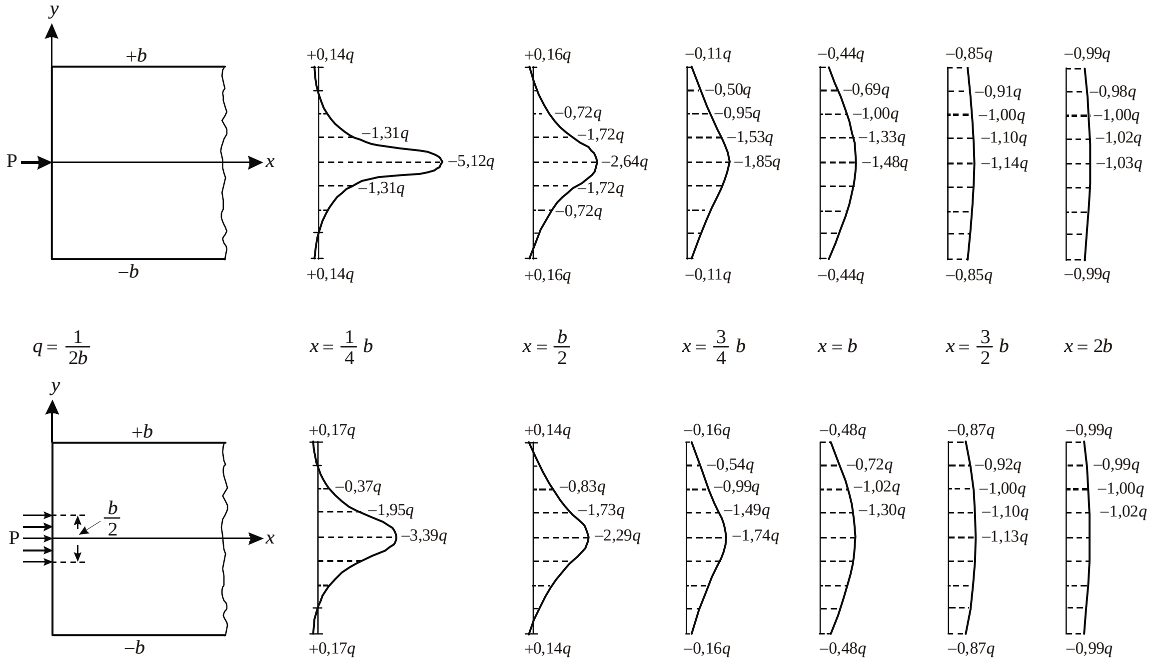


Figure 4.4 Surface of a body subjected to (a) Concentrated load and (b) Strip load of width *b/2*

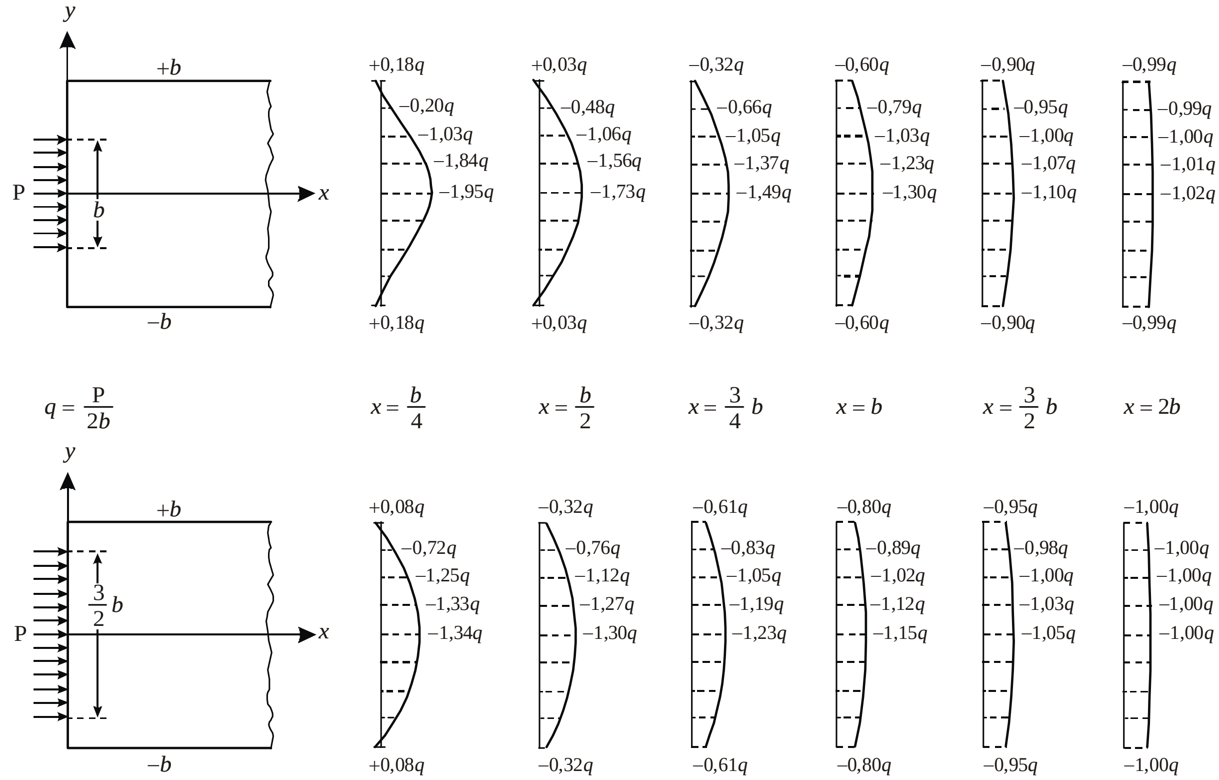


Figure 4.5 Surface of a body subjected to (a) Strip load of width *b* and (b) Strip load of width 1.5*b*

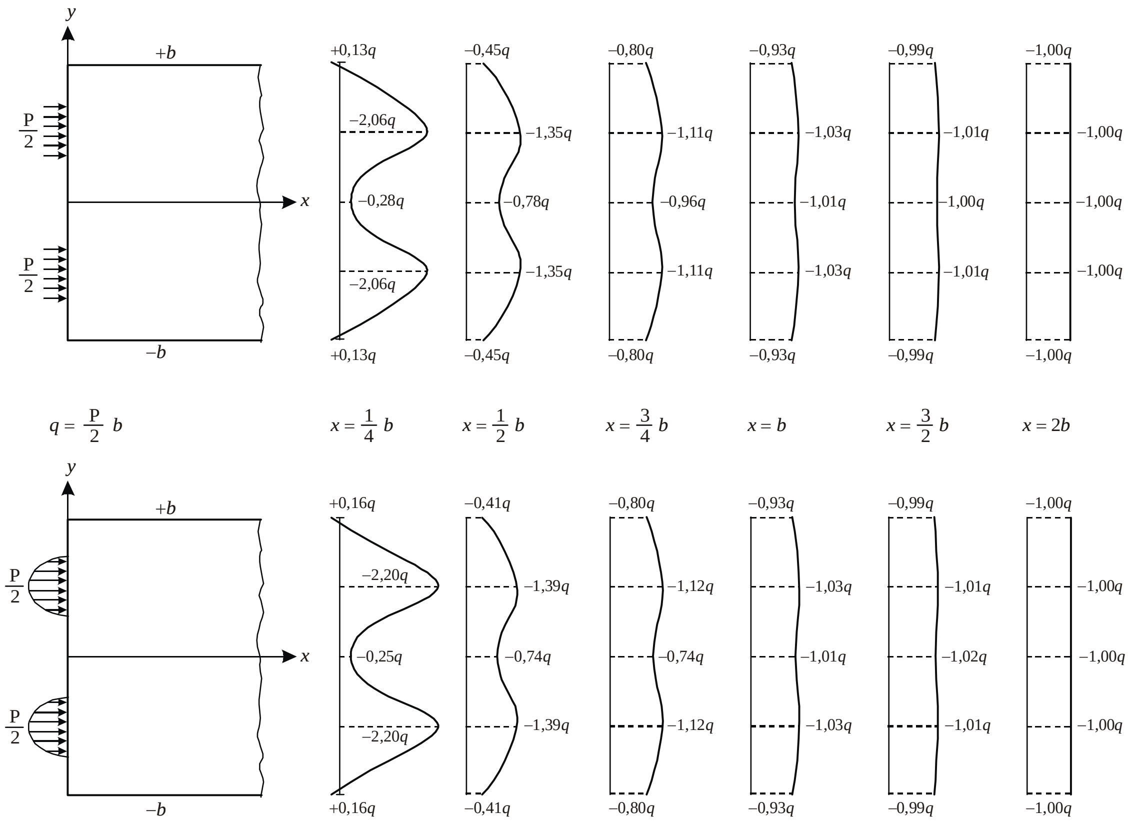


Figure 4.6 Surface of a body subjected to (a) Two strip load and (b) Inverted parabolic two strip loads

Figures 4.4, 4.5 and 4.6 demonstrate the distribution of stresses (q) in the body when subjected to various types of loading. In all the cases, the distribution of stress throughout the body is altered only near the regions of load application. However, the stress distribution is not altered at a distance $x = 2b$ irrespective of loading conditions.

4.2.5 EXISTENCE AND UNIQUENESS OF SOLUTION (UNIQUENESS THEOREM)

This is an important theorem in the theory of elasticity and distinguishes elastic deformations from plastic deformations. The theorem states that, for every problem of elasticity defined by a set of governing equations and boundary conditions, there exists one and only one solution. This means that “elastic problems have a unique solution” and two different solutions cannot satisfy the same set of governing equations and boundary conditions.

Proof

In proving the above theorem, one must remember that only elastic problems are dealt with infinitesimal strains and displacements. If the strains and displacements are not infinitesimal, the solution may not be unique.

Let a set of stresses $\sigma'_x, \sigma'_y, ...\tau'_{zx}$ represents a solution for the equilibrium of a body under surface forces *X, Y, Z* and body forces F_x, F_y, F_z. Then the equations of equilibrium and boundary conditions must be satisfied by these stresses, giving

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} + F_x = 0; \qquad (x, y, z)$$

and $\sigma'_x l + \tau'_{xy} m + \tau'_{xz} n = F_x; \qquad (x, y, z)$

where (x, y, z) means that there are two more equations obtained by changing the suffixes y for x and z for y, in a cyclic order.

Similarly, if there is another set of stresses $\sigma''_x, \sigma''_y, ...\tau''_{zx}$ which also satisfies the boundary conditions and governing equations we have,

$$\frac{\partial \sigma''_x}{\partial x} + \frac{\partial \tau''_{xy}}{\partial y} + \frac{\partial \tau''_{xz}}{\partial z} + x = 0; \qquad (x, y, z)$$

and $\sigma''_x l + \tau''_{xy} m + \tau''_{xz} n = F_x; \qquad (x, y, z)$

By subtracting the equations of the above set from the corresponding equations of the previous set, we get the following set,

$$\frac{\partial}{\partial x}(\sigma'_x - \sigma''_x) + \frac{\partial}{\partial y}(\tau'_{xy} - \tau''_{xy}) + \frac{\partial}{\partial z}(\tau'_{xz} - \tau''_{xz}) = 0; \quad (x, y, z)$$

and $(\sigma'_x - \sigma''_x)l + (\tau'_{xy} - \tau''_{xy})m + (\tau'_{xz} - \tau''_{xz})n = 0; \quad (x, y, z)$

In the same way it is shown that the new strain components $(\varepsilon'_x - \varepsilon''_x)$, $(\varepsilon'_y - \varepsilon''_y)$…. etc. also satisfy the equations of compatibility. A new solution $(\sigma'_x - \sigma''_x)$, $(\sigma'_y - \sigma''_y)$,….. $(\tau'_{xz} - \tau''_{xz})$ represents a situation where body forces and surface forces both are zero. The work done by these forces during loading is zero and it follows that the total strain energy vanishes, i.e.,

$$\int \int \int V_o \, dxdydz = 0$$

where $V_o = (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$

The strain energy per unit volume V_o is always positive for any combination of strains and stresses. Hence for the integral to be zero, V_o must vanish at all the points, giving all the stress components (or strain components) zero, for this case of zero body and surface forces.

Therefore $(\sigma'_x-\sigma''_x)=(\sigma'_y-\sigma''_y)=(\sigma'_z-\sigma''_z)= 0$

and $(\tau'_{xy}-\tau''_{xy}) = (\tau'_{yz}-\tau''_{yz}) = (\tau'_{zx}-\tau''_{zx}) = 0$

This shows that the set σ'_x, σ'_y, σ'_z,…. τ'_{zx} is identical to the set σ''_x, σ''_y, σ''_z, …. τ''_{zx} and there is one and only one solution for the elastic problem**.**

4.2.6 NUMERICAL EXAMPLES

Example 4.1

The following are the principal stress at a point in a stressed material. Taking $E = 210kN/mm^2$ and $\nu = 0.3$, calculate the volumetric strain and the Lame's constants.

$$\sigma_x = 200N/mm^2, \quad \sigma_y = 150N/mm^2, \quad \sigma_z = 120N/mm^2$$

***Solution*:** We have

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$= \frac{1}{210\times10^3}\left[200 - 0.3(150+120)\right]$$

$$\therefore \varepsilon_x = 5.67\times10^{-4}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right]$$

$$= \frac{1}{210\times10^3}\left[150 - 0.3(120+200)\right]$$

$$\therefore \varepsilon_y = 2.57\times10^{-4}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$= \frac{1}{210\times10^3}\left[120 - 0.3(200+150)\right]$$

$$\therefore \varepsilon_z = 7.14\times10^{-5}$$

Volumetric strain = $\varepsilon_v = \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)$

$$= 5.67\times10^{-4} + 2.57\times10^{-4} + 7.14\times10^{-5}$$

$$\therefore \varepsilon_v = 8.954\times10^{-3}$$

To find Lame's constants

We have, $G = \dfrac{E}{2(1+\nu)}$

$$G = \frac{210\times10^3}{2(1+0.3)}$$

$$\therefore G = 80.77\times10^3\, N/mm^2$$

$$\lambda = \frac{G(2G-E)}{(E-3G)}$$

$$= \frac{80.77\times10^3\left(2\times80.77\times10^3 - 210\times10^3\right)}{\left(210\times10^3 - 3\times80.77\times10^3\right)}$$

$$\therefore \lambda = 121.14\times10^3\, N/mm^2$$

Example 4.2

The state of strain at a point is given by

$\varepsilon_x = 0.001, \quad \varepsilon_y = -0.003, \quad \varepsilon_z = \gamma_{xy} = 0, \quad \gamma_{xz} = -0.004, \quad \gamma_{yz} = 0.001$

Determine the stress tensor at this point. Take $E = 210\times10^6 kN/m^2$, Poisson's ratio = 0.28. Also find Lame's constant.

***Solution*:** We have

$$G = \frac{E}{2(1+\nu)}$$

$$= \frac{210\times10^6}{2(1+0.28)}$$

$$\therefore G = 82.03\times10^6 kN/m^2$$

But $\lambda = \dfrac{G(2G-E)}{(E-3G)}$

$$= \frac{82.03\times10^6\left(2\times82.03\times16^6 - 210\times10^6\right)}{\left(210\times10^6 - 3\times82.03\times10^6\right)}$$

$$\therefore \lambda = 104.42\times10^6 kN/m^2$$

Now,

$$\sigma_x = (2G+\lambda)\varepsilon_x + \lambda\left(\varepsilon_y + \varepsilon_z\right)$$

$$= (2\times 82.03+104.42)10^6\times 0.001+104.42\times 10^6(-0.003+0)$$

$\therefore \sigma_x = -44780\, kN/m^2$

or $\sigma_x = -44.78 \quad MPa$

$$\sigma_y = (2G+\nu)\varepsilon_y + \lambda(\varepsilon_z+\varepsilon_x)$$

$$= (2\times 82.03+104.42)\times 10^6\times(-0.003)+104.42\times 10^6(0+0.001)$$

$\therefore \sigma_y = -701020\, kN/m^2$

or $\sigma_y = -701.02\, MPa$

$$\sigma_z = (2G+\lambda)\varepsilon_z + \lambda(\varepsilon_x+\varepsilon_y)$$

$$= (2\times 82.03+104.42)10^6(0)+104.42\times 10^6(0.001-0.003)$$

$\therefore \sigma_z = -208840\, kN/m^2$

or $\sigma_z = -208.84\, MPa$

$$\tau_{xy} = G\gamma_{xy}$$

$$= 82.03\times 10^6\times 0$$

$\therefore \tau_{xy} = 0$

$$\tau_{yz} = G\gamma_{yz} = 82.03\times 10^6\times 0.001 = 82030\, kN/m^2$$

or $\tau_{yz} = 82.03\, MPa$

$$\tau_{xz} = G\gamma_{xz} = 82.03\times 10^6\times(-0.004) = -328120\, kN/m^2$$

or $\tau_{xz} = -328.12\, MPa$

$\therefore$ The Stress tensor is given by

$$\sigma_{ij} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} = \begin{pmatrix} -44.78 & 0 & -328.12 \\ 0 & -701.02 & 82.03 \\ -328.12 & 82.03 & -208.84 \end{pmatrix}$$

Example 4.3

The stress tensor at a point is given as

$$\begin{pmatrix} 200 & 160 & -120 \\ 160 & -240 & 100 \\ -120 & 100 & 160 \end{pmatrix} kN/m^2$$

Determine the strain tensor at this point. Take $E = 210\times 10^6\, kN/m^2$ *and* $\nu = 0.3$

Solution: $\varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$

$$= \frac{1}{210\times10^6}\left[200 - 0.3(-240 + 160)\right]$$

$$\therefore \varepsilon_x = 1.067\times10^{-6}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right]$$

$$= \frac{1}{210\times10^6}\left[-240 - 0.3(160 + 200)\right]$$

$$\therefore \varepsilon_y = -1.657\times10^{-6}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$= \frac{1}{210\times10^6}\left[160 - 0.3(200 - 240)\right]$$

$$\therefore \varepsilon_z = 0.82\times10^{-6}$$

Now, $G = \dfrac{E}{2(1+\nu)} = \dfrac{210\times10^6}{2(1+0.3)} = 80.77\times10^6\, kN/m^2$

$$\tau_{xy} = G\gamma_{xy} = 80.77\times10^6\times\gamma_{xy}$$

$$\therefore \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{160}{80.77\times10^6} = 1.981\times10^{-6}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{100}{80.77\times10^6} = 1.24\times10^{-6}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{-120}{80.77\times10^6} = -1.486\times10^{-6}$$

Therefore, the strain tensor at that point is given by

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{pmatrix} = \begin{pmatrix} \varepsilon_x & \dfrac{\gamma_{xy}}{2} & \dfrac{\gamma_{xz}}{2} \\ \dfrac{\gamma_{xy}}{2} & \varepsilon_y & \dfrac{\gamma_{yz}}{2} \\ \dfrac{\gamma_{zx}}{2} & \dfrac{\gamma_{zy}}{2} & \varepsilon_z \end{pmatrix}$$

$$\therefore \varepsilon_{ij} = \begin{pmatrix} 1.067 & 0.9905 & -0.743 \\ 0.9905 & -1.657 & 0.62 \\ -0.743 & 0.62 & 0.82 \end{pmatrix}\times10^{-6}$$

Example 4.4

A rectangular strain rosette gives the data as below.

$\varepsilon_0 = 670\ micrometres/m$

$\varepsilon_{45} = 330\ micrometres/m$

$\varepsilon_{90} = 150\ micrometres/m$

Find the principal stresses σ_1 *and* σ_2 **if** $E = 2\times10^5 MPa, \nu = 0.3$

***Solution*:** We have

$$\varepsilon_x = \varepsilon_0 = 670\times10^{-6}$$

$$\varepsilon_y = \varepsilon_{90} = 150\times10^{-6}$$

$$\gamma_{xy} = 2\varepsilon_{45} - (\varepsilon_0 + \varepsilon_{90}) = 2\times330\times10^{-6} - \left(670\times10^{-6} + 150\times10^{-6}\right)$$

$$\therefore \gamma_{xy} = -160\times10^{-6}$$

Now, the principal strains are given by

$$\varepsilon_{max}\ or\ \varepsilon_{min} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

i.e., $\varepsilon_{max}\ or\ \varepsilon_{min} = \left(\frac{670+150}{2}\right)10^{-6} \pm \frac{1}{2}\sqrt{\left[(670-150)10^{-6}\right]^2 + \left(-160\times10^{-6}\right)^2}$

$$\therefore \varepsilon_{max}\ or\ \varepsilon_{min} = 410\times10^{-6} \pm 272.03\times10^{-6}$$

$$\therefore \varepsilon_{max} = \varepsilon_1 = 682.3\times10^{-6}$$

$$\varepsilon_{min} = \varepsilon_2 = 137.97\times10^{-6}$$

The principal stresses are determined by the following relations

$$\sigma_1 = \frac{(\varepsilon_1 + \nu\varepsilon_2)}{1-\nu^2}.E = \frac{(682.03 + 0.3\times137.97)10^{-6}}{1-(0.3)^2}\times2\times10^5$$

$$\therefore \sigma_1 = 159MPa$$

Similarly, $\sigma_2 = \frac{(\varepsilon_2 + \nu\varepsilon_1)}{1-\nu^2}.E = \frac{(137.97 + 0.3\times682.03)10^{-6}}{1-(0.3)^2}\times2\times10^5$

$$\therefore \sigma_2 = 75.3MPa$$